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Yang-Baxter σ-models and dS/AdS T-duality

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Abstract: We point out the existence of nonlinear σ-models on group manifolds which are left symmetric and right Poisson-Lie symmetric. We discuss the corresponding rich T-duality story with particular emphasis on two examples: the anisotropic principal chiral model and the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model. The latter has the de Sitter space as its (conformal) non-abelian dual.

Keywords: Sigma Models, Bosonic Strings, String Duality, Conformal Field Models in String Theory.

Contents

1. Introduction

This paper is devoted to the study of a particular class of σ-models living on group manifolds. The metric and the three-form H of such a model are *left*-invariant, however, the model is Poisson-Lie symmetric with respect to the *right* action of the group on itself. We shall refer to such σ-models as to the Yang-Baxter σ-models for reasons which will become clear in what follows.

The Yang-Baxter σ-models have interesting algebraic properties. First of all, they have a very rich T-duality story because they can be dualized either from the left (by the traditional non-abelian T-duality [1]) or from the right (by the Poisson-Lie T-duality [2]). They have also another remarkable feature: the phase space of a Yang-Baxter σ-model can be represented by two Drinfeld doubles *nonisomorphic* as *groups*. This leads e.g. to some insights about the structure of the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW theory.

In what follows, we describe the general formalism and then study the Yang-Baxter σ-models on a simple compact Lie group K and on the group AN in the Iwasawa decomposition of the complexified group $K^{\mathbb{C}} = KAN$. For $K^{\mathbf{C}} = \mathrm{SL}(2,\mathbb{C})$, the former example turns out to be the anisotropic principal chiral model (known to be integrable [3]) and the latter the (conformal) $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model describing the strings on the euclidean AdS_3 target. The AdS_3 model turns out to have the de Sitter background dS_3 (with an appropriate three-form field) as its Poisson-Lie T-dual.

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In section 2, we review the notion of the (right) Poisson-Lie symmetry of nonlinear σ-models and show how the model enjoying this property can be rewritten in a first order form by using the concept of the Drinfeld double. Then we show that the additional left symmetry generically means that the geometry of the target is given by the elements of the r-matrix defining the Poisson-Lie structure on the PL symmetry group. This gives the name to our models. We finish the section by considering the Poisson-Lie symmetry of (non-unitary) σ-models having a *real* euclidean action.

In section 3, we explicitely describe the Poisson-Lie symmetry of our two principal examples of the Yang-Baxter σ-models: the anisotropic principal chiral model on compact groups K and the $K^{\mathbf{C}}/K$ WZW theory. In section 4, we explain why the Poisson-Lie symmetry generically means the T-dualizability of the model and we describe the T-duals of the anisotropic model on SU(2) and of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model. Then we show how the rich Kac-Moody symmetry of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model translates under the duality to the same symmetry of the de Sitter model. We finish by short conclusions.

2. Poisson-Lie symmetry

In this section, we give a method for the construction of the Yang-Baxter σ-models.
1. As it is well-known, a two-dimensional non-linear σ-model is a field theory canonically associated to a metric (symmetric tensor) G_{ij} and a two-form (antisymmetric) tensor B_{ij} on some manifold M. Its action is given (in some local coordinates x^i) as

$$S=\frac{1}{2}\int d\sigma d\tau (G_{ij}(x)+B_{ij}(x))\partial_+x^i\partial_-x^j\equiv\frac{1}{2}\int d\sigma d\tau E_{ij}(x)\partial_+x^i\partial_-x^j\,, \tag{2.1}$$

where

$$\partial_\pm=\partial_\tau\pm\partial_\sigma\,,\qquad \xi^\pm=\frac{1}{2}(\tau\pm\sigma)$$

and τ and σ are respectively the worldsheet time and space coordinates. Suppose that there is a free right action of a Lie group G on M generated by vector fields $v_a(x)$ on M and consider the variation of the action (2.1) with respect to the G-transformations with the world-sheet dependent parameters $\varepsilon^a(\sigma,\tau)$:

$$\delta S\equiv S(x+\varepsilon^a v_a)-S(x)=\frac{1}{2}\int d\sigma d\tau\varepsilon^a(\mathcal{L}_{v_a}E_{ij})\partial_+x^i\partial_-x^j+\int J_a\wedge d\varepsilon^a\,.$$

Here $\mathcal{L}_{v_a}$ means the Lie derivative with respect to v_a and the worldsheet current one-forms J_a are

$$J_a=-v_a^i(x)E_{ij}(x)\partial_-x^jd\xi^-+v_a^i(x)E_{ji}(x)\partial_+x^jd\xi^+\,. \tag{2.2}$$

We say (following [2]) that the σ-model (2.1) is $\tilde{G}$-Poisson-Lie symmetric if

$$\delta S=\int\varepsilon^a\left(dJ^a-\frac{1}{2}\tilde{f}_a{}^{kl}J_k\wedge J_l\right), \tag{2.3}$$

where $\tilde{f}_a{}^{kl}$ are the structure constantes of $\tilde{\mathcal{G}}\equiv Lie(\tilde{G})$. The Poisson-Lie symmetry condition can be written also in terms of the σ-model tensor E_{ij} as follows

$$\mathcal{L}_{v_a}E_{ij}=-\tilde{f}_a{}^{kl}v_k^m v_l^n E_{mj}E_{in}\,. \tag{2.4}$$

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2. If the manifold M is itself the group G, there is a powerful method [2] of finding the right Poisson-Lie symmetric σ-models (i.e. of solving the condition (2.4)). It uses the crucial concept of the Drinfeld double[1] of G and $\tilde{G}$ and constructs the σ-model (2.1) by starting from its first order hamiltonian action written on the phase space. Recall that (for any dynamical system) the latter has always the form

$$S = \int (\theta - H d\tau) \,, \tag{2.5}$$

where $d\theta$ is the symplectic form and H a hamiltonian fonction.

The crucial statement [2] is as follows: the phase space of right $\tilde{G}$-Poisson-Lie symmetric σ-model on a Lie group target G can be identified with the loop group LD of the Drinfeld double D [4]. Thus a trajectory in the phase space is a map $l(\sigma,\tau)$ from the world-sheet into D where τ is interpreted as the evolution parameter. The first order action (2.5) of the right $\tilde{G}$-Poisson-Lie symmetric σ-model on a Lie group target G then reads

$$\begin{aligned} S = {} & \frac{1}{2}\int d\sigma d\tau \left(\partial_\tau l l^{-1}, \partial_\sigma l l^{-1}\right)_{\mathcal{D}} + \frac{1}{12}\int d^{-1}\left(dll^{-1} \overset{\wedge}{,} \left[dll^{-1} \overset{\wedge}{,} dll^{-1}\right]\right)_{\mathcal{D}} - \\ & -\frac{1}{2}\int d\sigma d\tau \left(\partial_\sigma l l^{-1}, \mathcal{E}\partial_\sigma l l^{-1}\right)_{\mathcal{D}} . \end{aligned} \tag{2.6}$$

The (chiral) WZW action in the first line correspond the term θ in (2.5) and the hamiltonian term is on the second line. It involves the operator $\mathcal{E} : \mathcal{D} \to \mathcal{D}$ fulfilling
1) $(\mathcal{E}x, y)_{\mathcal{D}} = (x, \mathcal{E}y)_{\mathcal{D}}$ ($\mathcal{E}$ is a selfadjoint);
2) $\mathcal{E}^2 = Id$ ($\mathcal{E}$ squares to the identity operator);
3) The G-dependent operator $\mathcal{J}\mathcal{E}_g\tilde{\mathcal{J}} : \tilde{\mathcal{G}} \to \mathcal{G}$ is invertible.
Here $\mathcal{J}(\tilde{\mathcal{J}}) : \mathcal{D} \to \mathcal{D}$ is the projector to $\mathcal{G}$ ($\tilde{\mathcal{G}}$) with the kernel $\tilde{\mathcal{G}}$ ($\mathcal{G}$) and $\mathcal{E}_g \equiv Ad_{g^{-1}}\mathcal{E}Ad_g$. It is important to stress that Ad_g means the adjoint action of an element $g \in G(\subset D)$ on $\mathcal{D}$.

3. Let us sketch the derivation of the second order lagrangian of the type (2.1) from the first order action (2.6). Our assumptions about the structure of the Drinfeld double D allow to decompose each trajectory $l(\sigma,\tau)$ as

$$l(\sigma,\tau) = g(\sigma,\tau)\tilde{h}(\sigma,\tau) \,, \qquad g \in G \,, \quad \tilde{h} \in \tilde{G} \,. \tag{2.7}$$

Inserting this decomposition into the action (2.6) and using the standard Polyakov-Wiegmann formula ($g^*, \tilde{h}^*$ mean the pull-backs of the maps $g, \tilde{h} : D \to D$)

$$\left(g\tilde{h}\right)^* c = g^* c + \tilde{h}^* c - d\left(g^*\left(l^{-1}dl\right) \overset{\wedge}{,} \tilde{h}^*\left(dll^{-1}\right)\right)_{\mathcal{D}} \;;$$
$$c = \frac{1}{6}\left(dll^{-1} \overset{\wedge}{,} \left[dll^{-1} \overset{\wedge}{,} dll^{-1}\right]\right)_{\mathcal{D}} \,,$$

[1]The Drinfeld double D is a $2n$-dimensional Lie group whose Lie algebra $\mathcal{D} = Lie(D)$, viewed as the linear space, can be decomposed into a direct sum of vector spaces $\mathcal{G}$ and $\tilde{\mathcal{G}}$, which are themselves maximally isotropic subalgebras of D with respect to a non-degenerate symmetric invariant bilinear form $(.,.)_{\mathcal{D}}$ on $\mathcal{D}$ [5, 6]. The Lie groups corresponding to the Lie algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ we denote as G and $\tilde{G}$ and we shall suppose, that each element l of D can be decomposed in unique way as the product $l = gb$, where $g \in G$ and $b \in \tilde{G}$. Moreover, the induced map $D \to G \times \tilde{G}$ should be a diffeomorphism. In the same way we suppose that there is a dual global smooth and unambiguous decomposition $D = \tilde{G}G$.

we obtain

$$S = \int \left(\partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g\right)_{\mathcal{D}} - \frac{1}{2}\int \left(g^{-1}\partial_\sigma g + \partial_\sigma \tilde{h}\tilde{h}^{-1}, \mathcal{E}_g\left(g^{-1}\partial_\sigma g + \partial_\sigma \tilde{h}\tilde{h}^{-1}\right)\right)_{\mathcal{D}}, \quad (2.8)$$

where and we tacitly suppose the measure $d\sigma d\tau$ present in the formula. Now we note that the expression (2.8) is gaussian in the $Lie(\tilde{G})$-valued variable $\partial_\sigma \tilde{h}\tilde{h}^{-1}$ which permits us to solve it away and to obtain a σ-model

$$S = \frac{1}{2}\int d\sigma d\tau \left(g^{-1}\partial_+ g, \left(\mathcal{J}\mathcal{E}_g\tilde{\mathcal{J}}\right)^{-1}(\mathcal{J} + \mathcal{J}\mathcal{E}_g\mathcal{J})\, g^{-1}\partial_- g\right)_{\mathcal{D}}. \quad (2.9)$$

In this derivation we have used the self-adjointness of $\mathcal{E}$ and the following consequences of the identity $\mathcal{E}^2 = \mathcal{E}_g^2 = Id$:

$$\left(\mathcal{J}\mathcal{E}_g\tilde{\mathcal{J}}\right)^{-1} = \tilde{\mathcal{J}}\mathcal{E}_g\mathcal{J} - \tilde{\mathcal{J}}\mathcal{E}_g\tilde{\mathcal{J}}\left(\mathcal{J}\mathcal{E}_g\tilde{\mathcal{J}}\right)^{-1}\mathcal{J}\mathcal{E}_g\mathcal{J}\,;$$
$$\left(\mathcal{J}\mathcal{E}_g\tilde{\mathcal{J}}\right)^{-1}\mathcal{J}\mathcal{E}_g\mathcal{J} = -\tilde{\mathcal{J}}\mathcal{E}_g\tilde{\mathcal{J}}\left(\mathcal{J}\mathcal{E}_g\tilde{\mathcal{J}}\right)^{-1}. \quad (2.10)$$

We invite the reader to check, that, indeed, the resulting σ-model (2.9) is right $\tilde{G}$-Poisson-Lie symmetric. It is also immediate that the left invariance of (2.9) takes place if $\mathcal{E}_g = \mathcal{E}$, i.e. if $\mathcal{E}$ is a G-invariant operator on $\mathcal{D}$.

Conclusion: let D be the Drinfeld double of the group G and $\mathcal{E}$ the self-adjoint, unipotent, G-invariant operator on $\mathcal{D}$, such that $\mathcal{J}\mathcal{E}\tilde{\mathcal{J}}$ is invertible. Then the action

$$S = \frac{1}{2}\int d\sigma d\tau \left(g^{-1}\partial_+ g, \tilde{R}g^{-1}\partial_- g\right)_{\mathcal{D}}, \quad (2.11)$$

with

$$\tilde{R} \equiv \left(\mathcal{J}\mathcal{E}\tilde{\mathcal{J}}\right)^{-1}(\mathcal{J} + \mathcal{J}\mathcal{E}\mathcal{J})$$

defines the Yang-Baxter σ-model on G.
4. The reader might wish to see a direct demonstration that the Yang-Baxter σ-model, (2.11) is right Poisson-Lie symmetric. For this, let us first calculate the current forms J_a according to (2.2):

$$J_a T^a = -\tilde{R}g^{-1}\partial_- g d\xi^- + \tilde{R}^\dagger g^{-1}\partial_+ g d\xi^+, \quad (2.12)$$

where T^a is the basis of $\tilde{\mathcal{G}}$ in which the structure constants are $\tilde{f}_a{}^{kl}$. It is easy to see that the hamiltonian field equations derived from the first order action (2.6) read

$$(Id \mp \mathcal{E})\partial_\pm ll^{-1} = 0\,.$$

Using the decomposition $l = g\tilde{h}$, the G-invariance of $\mathcal{E}$ an the the fact that $\tilde{R}^\dagger = (\mathcal{J}\mathcal{E}\tilde{\mathcal{J}})^{-1}\times(\mathcal{J} - \mathcal{J}\mathcal{E}\mathcal{J})$, we obtain

$$\partial_-\tilde{h}\tilde{h}^{-1} = -\tilde{R}g^{-1}\partial_- g\,, \qquad \partial_+\tilde{h}\tilde{h}^{-1} = \tilde{R}^\dagger g^{-1}\partial_+ g \quad (2.13)$$

and from (2.13) it clearly follows that the field equations of the Yang-Baxter σ-model have the zero-curvature form (2.3):

$$dJ_a = \frac{1}{2}\tilde{f}_a{}^{kl} J_k \wedge J_l\,.$$

5. Now we turn to the explication of the title of this article. Suppose that the operator $\tilde{R} : \mathcal{G} \to \tilde{\mathcal{G}}$ has an inverse $R : \tilde{\mathcal{G}} \to \mathcal{G}$. The properties of $\mathcal{E}$ then imply the following identity

$$\mathcal{J} Ad_g \tilde{\mathcal{J}} Ad_{g^{-1}} \tilde{\mathcal{J}} = \mathcal{J} Ad_g \mathcal{J} R \tilde{\mathcal{J}} Ad_{g^{-1}} \tilde{\mathcal{J}} - R. \qquad (2.14)$$

The g-dependent operator from $\tilde{\mathcal{G}}$ into $\mathcal{G}$ on the l.h.s. of (2.14) deserves a special name; we call it Π_g:

$$\Pi_g = \mathcal{J} Ad_g \tilde{\mathcal{J}} Ad_{g^{-1}} \tilde{\mathcal{J}}\,. \qquad (2.15)$$

In fact, Π_g turns out to be the Poisson-Lie structure on G induced by the double D [4]. For completeness, we write the corresponding Poisson-Lie bracket of two functions ϕ, ψ on G:

$$\{\phi, \psi\}_G = (\nabla_L \phi, \Pi_g \nabla_L \psi)_{\mathcal{D}}\,. \qquad (2.16)$$

Here the differential operator ∇_L is considered to live in $\tilde{\mathcal{G}}$. It is defined by

$$(\nabla_L, x)_{\mathcal{D}} = \nabla_L^x\,,$$

where ∇_L^x is the differential operator acting on functions on G and corresponding to the left infinitesimal action of the Lie algebra element $x \in \mathcal{G}$.

Now note that the relations (12,13) mean that the Poisson-Lie bracket (2.16) on G can be rewritten in the following form

$$\{\phi, \psi\}_G = (\nabla_R \phi, R \nabla_R \psi)_{\mathcal{D}} - (\nabla_L \phi, R \nabla_L \psi)_{\mathcal{D}}\,,$$

where $\nabla_{R(L)}$ corresponds to the right (left) infinitesimal action on G. It is now clear that R gives rise to an element $r \in \mathcal{G} \otimes \mathcal{G}$ such that the Poisson-Lie bracket can be rewritten in the quasitriangular way:

$$\{\phi, \psi\}_G = \langle r, \nabla_R \phi \otimes \nabla_R \psi \rangle - \langle r, \nabla_L \phi \otimes \nabla_L \psi \rangle\,.$$

Here $\langle ., . \rangle$ denotes the canonical pairing between the vector space and its dual. Since the Poisson bracket is antisymmetric, we see that the symmetric part r_S of r is G-invariant. The Jacobi identity, in turn, then implies the (modified) Yang-Baxter equations for the antisymmetric part r_A of r:

$$Ad_G [r_A, r_A]_{Sch} = 0\,,$$

where $[., .]_{Sch}$ is the algebraic Schouten bracket.

Thus we observe that the choice of the G-invariant selfadjoint unipotent $\mathcal{E}$ with invertible $\tilde{R}$ gives rise to the r-matrix encoding the (quasitriangular) Poisson-Lie structure on G. This explains the adjective "Yang-Baxter" which we have attributed to the left-invariant and right Poisson-Lie symmetric σ-models (2.11). By a slight abuse of the terminology, we stick to this name also in cases where $\tilde{R}$ is not invertible.

6. It is well-known that the unitary σ-models do not have a real action on the euclidean world-sheet. Nevertheless, there is a class of important (though non-unitary) euclidean models which do have the real action. The most prominent example of such a theory is the $\mathrm{SL}(2, \mathbb{C}) / \mathrm{SU}(2)$ WZW model describing strings on the euclidean AdS_3. Here we define the

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notion of the Poisson-Lie symmetry for the class of the euclidean σ-models with the real action

$$S = \frac{1}{2}\int d\sigma d\tau E_{ij}(x)\partial_z x^i \partial_{\bar{z}} x^j ,$$

where

$$\partial_z = \partial_\tau + i\partial_\sigma , \qquad \partial_{\bar{z}} = \partial_\tau - i\partial_\sigma .$$

The reality of the lagrangian require the matrix E_{ij} to be hermitean ($E_{ij} = \bar{E}_{ji}$). The definition (2.2) of the current one-forms J_a now gives

$$J_a = iv^i_a(x)E_{ji}(x)\partial_z x^j dz - iv^i_a(x)E_{ij}(x)\partial_{\bar{z}} x^j d\bar{z} ,$$

where

$$z = \frac{1}{2}(\tau - i\sigma), \quad \bar{z} = \frac{1}{2}(\tau + i\sigma) .$$

With this notation, the condition of the right euclidean Poisson-Lie symmetry is given by the same expression (2.3) as in the Minkowski case. The euclidean analogue of (2.4) reads

$$\mathcal{L}_{v_a} E_{ij} = -i\tilde{f}_a{}^{kl} v^m_k v^n_l E_{mj} E_{in} . \tag{2.17}$$

In the euclidean case, it is also easy to generate right Poisson-Lie symmetric models from the action (2.6) on the double. The only modification is that now $\mathcal{E}$ in (2.6) has to fulfil $\mathcal{E}^2 = -Id$. In particular, the resulting Yang-Baxter σ-model on the euclidean worldsheet has the lagrangian

$$L = \frac{1}{2}\int d\sigma d\tau \left(g^{-1}\partial_z g, \tilde{R}_{eu} g^{-1}\partial_{\bar{z}} g\right)_{\mathcal{D}} \tag{2.18}$$

with

$$\tilde{R}_{eu} = \left(\mathcal{J}\mathcal{E}\tilde{\mathcal{J}}\right)^{-1} (\mathcal{J} - i\mathcal{J}\mathcal{E}\mathcal{J}) .$$

3. Examples

3.1 Anisotropic principal chiral model

1. Now it is time to give examples. For the group G we take a simple connected and simply connected compact group K and for its Drinfeld double D its complexification $D = K^{\mathbf{C}}$. So, for instance, the double of SU(2) is SL(2, $\mathbf{C}$). The invariant bilinear form on $\mathcal{D} = Lie(D)$ is given by

$$(x, y)_{\mathcal{D}} = \mathrm{Im} T(x, y) ,$$

or, in other words, it is just the imaginary part of the Killing-Cartan form $T(.,.)$ normalized in the way that the square of the lenght of the longest root is equal to two. Since G is the real form of $K^{\mathbb{C}}$, clearly, the imaginary part of $T(x, y)$ vanishes if $x, y \in \mathcal{G}$. Hence, $G = K$ is isotropically embedded in $K^{\mathbb{C}}$.

The dual group $\tilde{G}$ coincides with the so called AN group in the Iwasawa decomposition of $K^{\mathbb{C}}$:

$$K^{\mathbb{C}} = KAN . \tag{3.1}$$

For the groups SL$(n, \mathbf{C})$ the group AN can be identified with upper triangular matrices of determinant 1 and with positive real numbers on the diagonal. In general, the elements of AN can be uniquely represented by means of the exponential map as follows

$$b = \mathrm{e}^{\phi}\exp[\Sigma_{\alpha>0}v_\alpha E_\alpha] \equiv \mathrm{e}^{\phi}n\,. \tag{3.2}$$

Here α's denote the roots of $\mathcal{K}^{\mathbb{C}}$, v_α are complex numbers and ϕ is an hermitean element[2] of the Cartan subalgebra of $\mathcal{K}^{\mathbb{C}}$. Loosely said, A is the "noncompact part" of the complex maximal torus of $K^{\mathbb{C}}$. The isotropy of the Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G} = AN$ follows from (3.2); the fact that $\mathcal{G}$ and $\tilde{\mathcal{G}}$ generate together the Lie algebra $\mathcal{D}$ of the whole double is evident from (3.1).

2. We wish to construct (minkowskian) Yang-Baxter σ-models on the compact target K. Consider the following $\mathbb{R}$-linear operator $\mathcal{E}_1 : \mathcal{D} \to \mathcal{D}$:

$$\mathcal{E}_1 x = \mathcal{I}x^\dagger\,. \tag{3.3}$$

Here $\mathcal{I}$ it is the multiplication by the imaginary unit on $\mathcal{K}^{\mathbb{C}}$(viewed as the *real* Lie algebra). It is easy to see that $\mathcal{E}_1^2 = Id$ and $\mathcal{E}_1$ is selfadjoint, i.e. for any $x, y \in \mathcal{D}$ it holds $(\mathcal{E}_1 x, y)_\mathcal{D} = (x, \mathcal{E}_1 y)_\mathcal{D}$. Moreover, $\mathcal{E}_1$ is K-invariant as

$$\mathcal{E}_1 Ad_k x = \mathcal{E}_1\left(kxk^\dagger\right) = \mathcal{I}kx^\dagger k^\dagger = Ad_k\mathcal{E}_1 x\,.$$

Thus, $P_1 = \frac{1}{2}(1+\mathcal{E}_1)$ is the K-invariant idempotent. In fact, it belongs to a one-parametric family of the invariant idempotents denoted P_α who project on the subspaces $(\alpha Id - \mathcal{I})\mathcal{K} \subset \mathcal{K}^{\mathbb{C}}, \alpha > 0$ and whose kernels are $(\alpha Id + \mathcal{I})\mathcal{K}$. Since the kernel is clearly orthogonal to the image of the projector, the idempotent P_α is selfadjoint. Moreover P_α is K-invariant, since both its image and kernel are K-invariant. Finally, it can be easily verified that $\mathcal{J}(2P_\alpha - Id)\tilde{\mathcal{J}}$ is invertible. Thus the choice $\mathcal{E}_\alpha = 2P_\alpha - Id$ gives a one parameter family of the Yang-Baxter σ-model on G with the lagrangian given by (2.11).

3. The reader may wish to visualize more explicitely what the Yang-Baxter σ-model is about for the simplest case of K = SU(2). For this, it is convenient to choose a basis $T^i = \frac{1}{2}i\sigma^i, i = 1, 2, 3$ in $su(2) \equiv Lie(\mathrm{SU}(2))$, where σ^i are the standard Pauli matrices:

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T^3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.4}$$

The dual basis in $Lie(AN)$ is then

$$t_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}, \qquad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

[2]Recall that the hermitean element of any complex simple Lie algebra $\mathcal{K}^{\mathbb{C}}$ is an eigenvector of the involution which defines the compact real form $\mathcal{K}$; the corresponding eigenvalue is (-1) . This involution comes from the group involution $g \to (g^{-1})^\dagger$. The anti-hermitean elements that span the compact real form are eigenvectors of the same involution with the eigenvalue equal to 1. For elements of $sl(n, \mathrm{C})$ Lie algebra, the hermitean element is indeed a hermitean matrix in the standard sense of this term.

Here by the duality we mean the following relation between the two basis: 1

$$(T^i, t_j)_{\mathcal{D}} = \delta^i{}_j\,.$$

It is not difficult to evaluate the operator $\tilde{R}_\alpha = (\mathcal{J}\mathcal{E}_\alpha\tilde{\mathcal{J}})^{-1}(\mathcal{J}+\mathcal{J}\mathcal{E}_\alpha\mathcal{J})$, where $\mathcal{E}_\alpha = 2P_\alpha - 1$. It is given by

$$\tilde{R}_\alpha T^1 = \frac{(\alpha t_1 - t_2)}{2(\alpha^2+1)}\,, \qquad \tilde{R}_\alpha T^1 = \frac{(\alpha t_2 + t_1)}{2(\alpha^2+1)}\,, \qquad \tilde{R}_\alpha T^3 = \frac{1}{2\alpha} t_3\,. \tag{3.6}$$

The lagrangian (xy) of the Yang-Baxter σ-model on SU(2) can be now written as

$$\begin{aligned} L = {}& \frac{1}{4\alpha}(k^{-1}\partial_+ k)_3(k^{-1}\partial_- k)_3 + \frac{\alpha}{4(\alpha^2+1)}(k^{-1}\partial_+ k)_1(k^{-1}\partial_- k)_1 + \\ & + \frac{\alpha}{4(\alpha^2+1)}(k^{-1}\partial_+ k)_2(k^{-1}\partial_- k)_2\,. \end{aligned} \tag{3.7}$$

Here we have decomposed $k^{-1}dk$ as $(k^{-1}dk)_j T^j$ and neglected the total derivatives. By the way, the model (3.7) is known to be integrable and it is usually referred to as the anisotropic principal chiral model. It is interesting to note that we have discovered a new algebraic structure of this model which will probably contribute to a better understanding of its quantum properties.

It is also important to remark, that the obtained model (3.7) is singular for $\alpha = 0$ and $\alpha \to \infty$. Indeed, for these values of the parametres the idempotents P_α are not well defined (the kernel would be the same as the image!) The singularities can be very naturally interpreted also from the geometrical point of view. Indeed, for $\alpha \to 0$, the ratio of the (anisotropic) third coefficient with respect to the first (or second) one goes to infinity. This is the extreme anisotropic limit where the σ-model geometry degenerates in one direction. In the case $\alpha \to \infty$ this ratio approaches 1, which corresponds the standard (isotropic) principal chiral model. However, the singularity shows up in the overall coefficient going to zero.

3.2 $K^{\mathbb{C}}/K$ WZW model

1. Our next example has the real action in the euclidean signature. Its underlying Drinfeld double is the same group $K^{\mathbb{C}}$ as in the previous one but the role of the groups G and $\tilde{G}$ gets reversed. This means that the $K^{\mathbb{C}}/K$ WZW model lives on the group $G = AN$ and it is right Poisson-Lie symmetric with respect to the dual group $\tilde{G} = K$.

The first order action of the $K^{\mathbb{C}}/K$ WZW model is given by the expression (2.6) with $\mathcal{E} = \mathcal{I}$ (cf.$\tilde{(}$3.3)). Clearly, $\mathcal{I}$ is selfadjoint (yes, it sometimes happens that the multiplication by the imaginary unit is a selfadjoint operator…), $\mathcal{I}$ is $K^{\mathbb{C}}$-invariant, $\mathcal{I}^2 = -Id$ and $\mathcal{J}\mathcal{I}\tilde{\mathcal{J}}$ is invertible. Thus $K^{\mathbb{C}}/K$ WZW theory is the Yang-Baxter σ-model with the second order real euclidean lagrangian (2.18) and $\tilde{R}_{eu} = (\mathcal{J}\mathcal{I}\tilde{\mathcal{J}})^{-1}(\mathcal{J} - i\mathcal{J}\mathcal{I}\mathcal{J})$.

2. As in the previous example, we give the detailed description of the case $K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$. We shall work with the same basis of $\mathcal{D}$ as before but now we remember that the basis t_a is that of $\mathcal{G} = Lie(AN)$ and the dual one T^a that of $\tilde{\mathcal{G}} = \mathcal{K}$. The operator $\mathcal{I}$ reads

$$\begin{aligned} \mathcal{I}t_1 &= -t_2\,, \quad \mathcal{I}t_2 = t_1\,, \quad \mathcal{I}t_3 = 2T^3\,; \\ \mathcal{I}T^1 &= -\frac{1}{2}t_1 + T^2\,, \quad \mathcal{I}T^2 = -\frac{1}{2}t_2 - T^1\,, \quad \mathcal{I}T^3 = -\frac{1}{2}t_3\,. \end{aligned} \tag{3.8}$$

It can be directly checked that $\mathcal{I}$ is selfadjoint with respect to the form $(.,.)_{\mathcal{D}}$, it is $\mathrm{SL}(2,\mathbb{C})$-invariant and it evidently holds $\mathcal{I}^2 = -Id$. With the choice $\mathcal{E} = \mathcal{I}$, the first order action (2.6) takes the form:

$$S = \frac{1}{2}\int d\sigma d\tau \mathrm{ImTr}\left(\partial_\tau ll^{-1}\partial_\sigma ll^{-1}\right) + \frac{1}{12}\int d^{-1}\mathrm{ImTr}\left(dll^{-1}\wedge\left[dll^{-1} \overset{\wedge}{,} dll^{-1}\right]\right) - \\ -\frac{1}{2}\int d\sigma d\tau \mathrm{ReTr}\left(\partial_\sigma ll^{-1}\partial_\sigma ll^{-1}\right) \tag{3.9}$$

or, even more simply,

$$S = \frac{1}{2}\int d\sigma d\tau \mathrm{ImTr}\left(\partial_{\bar{z}} ll^{-1}\partial_\sigma ll^{-1}\right) + \frac{1}{12}\int d^{-1}\mathrm{ImTr}\left(dll^{-1}\wedge\left[dll^{-1} \overset{\wedge}{,} dll^{-1}\right]\right). \tag{3.10}$$

In words: the first order action of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model can be written as a sort of semi-chiral WZW model on the complex group $\mathrm{SL}(2,\mathbb{C})$.

3. The resulting second order lagrangian on $G = AN$ can be obtained by decomposing

$$l = bk\,, \qquad b \in AN, k \in \mathrm{SU}(2)\,, \tag{3.11}$$

and eliminating $\partial_\sigma kk^{-1}$. The result is (cf.~(2.18)) written as

$$\tilde{L} = \frac{1}{2}\left(b^{-1}\partial_z b, \tilde{R}_{eu} b^{-1}\partial_{\bar{z}} b\right)_{\mathcal{D}}, \tag{3.12}$$

with

$$\tilde{R}_{eu} = \left(\mathcal{J}\mathcal{I}\tilde{\mathcal{J}}\right)^{-1}(\mathcal{J} - i\mathcal{J}\mathcal{I}\mathcal{J})\,.$$

From (3.4), (3.5) and (3.8), it is easy to calculate the operator $\tilde{R}_{eu} : \mathcal{G} \to \tilde{\mathcal{G}}$:

$$\tilde{R}_{eu}t_1 = -2\left(T^1 + iT^2\right), \qquad \tilde{R}_{eu}t_2 = -2\left(T^2 - iT^1\right), \qquad \tilde{R}_{eu}t_3 = -2T^3\,. \tag{3.13}$$

Using the following parametrization of the group AN:

$$b = \begin{pmatrix} e^\phi & 2e^{-\phi}(u_1 - iu_2) \\ 0 & e^{-\phi} \end{pmatrix}, \tag{3.14}$$

we calculate $b^{-1}db = (b^{-1}db)^j t_j$:

$$(b^{-1}db)^1 = e^{-2\phi}du_1, (b^{-1}db)^2 = e^{-2\phi}du_2\,, \qquad (b^{-1}db)^3 = d\phi\,. \tag{3.15}$$

Putting together (3.12), (3.13) and (3.15) and setting $u = u_1 + iu_2$, we obtain that the Yang-Baxter lagrangian (3.12) gives the action of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model (cf. [7, 8]):

$$S = -\int d\sigma d\tau \left(\partial_z\phi\partial_{\bar{z}}\phi + e^{-4\phi}\partial_z u\partial_{\bar{z}}\bar{u}\right). \tag{3.16}$$

Up to an unimportant overall normalization, the background metric and H-field of the model (xy) are

$$ds^2_{AdS} = d\phi d\phi + e^{-4\phi}dud\bar{u}\,, \qquad H = -2e^{-4\phi}d\phi\wedge du\wedge d\bar{u}\,. \tag{3.17}$$

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If we set $\tilde{R}_{ab} = (t_a, \tilde{R}_{eu} t_b)_{\mathcal{D}}$, the condition (2.17) expressing the right Poisson-Lie symmetry of (3.12) can be rewritten as

$$f_{ca}{}^{n}\tilde{R}_{nb} + f_{cb}{}^{n}\tilde{R}_{an} = i\tilde{f}_{c}{}^{kl}\tilde{R}_{kb}\tilde{R}_{al}\,. \tag{3.18}$$

Here $f_{ca}{}^{n}$ are the structure constants of $Lie(AN)$ in the basis (3.5) and $\tilde{f}_{c}{}^{kl}$ of $su(2)$ in the basis (3.4). It is easy to verify that our operator $\tilde{R}_{eu}$ given by (3.13) satisfies (3.18).

The $su(2)$-valued Poisson-Lie current J is given by

$$\begin{aligned} J &= i\tilde{R}^{\dagger}g^{-1}\partial_z g dz - i\tilde{R}g^{-1}\partial_{\bar{z}} g d\bar{z} = \\ &= -2ie^{-2\phi}\partial_z u(T^1 - iT^2)dz - 2i\partial_z\phi T^3 dz + 2ie^{-2\phi}\partial_{\bar{z}}\bar{u}(T^1 + iT^2)dz + 2i\partial_{\bar{z}}\phi T^3 d\bar{z}\,. \end{aligned}$$

It is easy to verify that the zero curvature condition $dJ = J \wedge J$ is equivalent to the field equations of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model:

$$\begin{aligned} \partial_z\partial_{\bar{z}}\phi + 2e^{-4\phi}\partial_z u \partial_{\bar{z}}\bar{u} &= 0\,, \\ \partial_z(e^{-4\phi}\partial_{\bar{z}}\bar{u}) = 0\,, \qquad \partial_{\bar{z}}(e^{-4\phi}\partial_z u) &= 0\,. \end{aligned}$$

Note: the Yang-Baxter property of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW-model leads naturally to two different representations of the phase space of the theory: either as the complex loop group $L\,\mathrm{SL}(2,\mathbb{C})$ or as the loop group of the cotangent bundle of AN. The latter representation (which is naturally induced by the left invariance) is well-known and discussed in literature (cf. [8]). Here we advance the former one. Its advantage may consist in the fact that the first order action is the WZW action on the *complex* group, hence it allows the Wakimoto free field representation (cf. [6]). Of course, the symplectic form in the cotangent bundle representation can also be written in Darboux coordinates [8], but the hamiltonian is not gaussian. In the loop group $L\,\mathrm{SL}(2,\mathbb{C})$ representation, the whole action (not only the symplectic form part) can be written in the gaussian way.

4. Poisson-Lie T-duality

1. The important structural feature of the $\tilde{G}$-Poisson-Lie symmetric σ-models on G is their generic T-dualizability, i.e. (if a condition 4 below is fulfilled then) there is the dual σ-model on $\tilde{G}$ which is moreover G-Poisson-Lie symmetric. In general, T-duality is a dynamical equivalence between two (or more) σ-models living on targets with different geometry and/or topology. The Poisson-Lie T-duality is the particular form of the T-duality, in which the common dynamics of all equivalent σ-models is given by the first order action (2.6) living on the Drinfeld double D. Why the action (2.6) can encode more than one σ-model (2.9)? The point is simple, set $\mathcal{E}_{\tilde{g}} = Ad_{\tilde{g}^{-1}}\mathcal{E}Ad_{\tilde{g}}$ and suppose that the conditions 1)-3) on page 4 are supplemented by another one:
4) the $\tilde{G}$-dependent operator $\tilde{\mathcal{J}}\mathcal{E}_{\tilde{g}}\mathcal{J}$ is invertible.
Then we may insert into the action (2.6) the dual counterpart of the decomposition (2.7):

$$l(\sigma,\tau) = \tilde{g}(\sigma,\tau)h(\sigma,\tau)\,, \qquad \tilde{g} \in \tilde{G}\,, \quad h \in G\,.$$

Mimicking the derivation after eq. (2.7), we eliminate the quantity $\partial_\sigma h h^{-1}$ and obtain the σ-model living on $\tilde{G}$:

$$\tilde{S} = \frac{1}{2}\int \left(\tilde{g}^{-1}\partial_+\tilde{g}, \left(\tilde{\mathcal{J}}\mathcal{E}_{\tilde{g}}\mathcal{J}\right)^{-1}\left(\tilde{\mathcal{J}} + \tilde{\mathcal{J}}\mathcal{E}_{\tilde{g}}\tilde{\mathcal{J}}\right)\tilde{g}^{-1}\partial_-\tilde{g}\right)_{\mathcal{D}}. \tag{4.1}$$

2. The duality between the σ-models (2.9) and (4.1) is the original Poisson-Lie T-duality described in [2]. It was later generalized in [9] by noting, that one can sometimes extract from the first order action (2.6) also σ-models whose targets are not Lie group manifolds. Indeed, suppose that there is a subgroup M of the double D, whose Lie algebra $\mathcal{M}$ is maximally isotropic (it has thus the same dimension as $\mathcal{G}$ or $\tilde{\mathcal{G}}$). The group D is then called the Manin double[3] of M [10]. Consider then the right coset D/M and parametrize[4] it by the elements f of D. With this parametrization of D/M, we may parametrize the surface $l(\tau,\sigma)$ in the double as follows

$$l(\tau,\sigma) = f(\tau,\sigma)m(\tau,\sigma)\,, \quad m \in M\,. \tag{4.2}$$

The action (2.6) then becomes

$$\begin{aligned} S = {} & \frac{1}{2}\int \left(f^{-1}\partial_\tau f, f^{-1}\partial_\sigma f\right)_{\mathcal{D}} + \frac{1}{12}\int d^{-1}\left(df f^{-1}, [df f^{-1}, df f^{-1}]\right)_{\mathcal{D}} + \\ & + \int \left(\partial_\sigma m m^{-1}, f^{-1}\partial_\tau f\right)_{\mathcal{D}} - \frac{1}{2}\int \left(f^{-1}\partial_\sigma f + \partial_\sigma m m^{-1}, \mathcal{E}_f\left(f^{-1}\partial_\sigma f + \partial_\sigma m m^{-1}\right)\right)_{\mathcal{D}}, \end{aligned} \tag{4.3}$$

where $\mathcal{E}_f = Ad_{f^{-1}}\mathcal{E}Ad_f$ and we tacitly suppose the measure $d\sigma d\tau$ present in the formula. Now we note that the expression (4.3) is gaussian in the $\mathcal{M}$-valued variable $\partial_\sigma m m^{-1}$. In order to solve it away, we do not use a projector on $\mathcal{M}$ (an analogue of $\mathcal{J}$ or $\tilde{\mathcal{J}}$) since there is no canonically given kernel. The most useful strategy is to pick up some basis S^a in $\mathcal{M}$, write $\partial_\sigma m m^{-1} = \mu_a S^a$ and integrate away μ_a. This gives

$$\begin{aligned} S = {} & \frac{1}{4}\int d\sigma d\tau\left(\partial_+ f f^{-1}, \partial_- f f^{-1}\right)_{\mathcal{D}} + \frac{1}{12}\int d^{-1}\left(df f^{-1}, [df f^{-1}, df f^{-1}]\right)_{\mathcal{D}} + \\ & + \frac{1}{2}\int d\sigma d\tau\left(f^{-1}\partial_+ f, S^a - \mathcal{E}_f S^a\right)_{\mathcal{D}}\left(A_f^{-1}\right)_{ab}\left(S^b, f^{-1}\partial_- f\right)_{\mathcal{D}}, \end{aligned} \tag{4.4}$$

where

$$A_f^{ab} = \left(S^a, \mathcal{E}_f S^b\right)_{\mathcal{D}}. \tag{4.5}$$

We thus conclude that a σ-model (4.4) can be associated to every maximally isotropic subalgebra of $\mathcal{D}$ provided the matrix A_f is invertible. The target of this σ-model is the coset D/M. It turns out [9] that the maximally isotropic subgroups M and M' related by a conjugation by a fixed element of the group D give rise respectively to the same

[3]Note a subtlety: we do not require the existence of the complementary dual group $\tilde{M}$ whose Lie algebra $\tilde{\mathcal{M}}$ would be maximally isotropic and $\mathcal{D}$ could be represented as the direct sum of the vector spaces $\mathcal{M}$ and $\tilde{\mathcal{M}}$. If such a complementary group existed we would say that D is the Drinfeld double of M.

[4]If there exists no global section of this fibration, we can choose several local sections covering the whole base space D/M.

σ-models. The problem of classifying the Poisson-Lie dual (or plural) models thus reduces to the search of all maximally isotropic subalgebras of $\mathcal{D}$ up to conjugation [9].

We finish this section by giving the euclidean analogue of the formula (4.4), taking place for $\mathcal{E}^2 = -Id$:

$$S = -\frac{i}{4}\int d\sigma d\tau \left(\partial_z f f^{-1}, \partial_{\bar{z}} f f^{-1}\right)_{\mathcal{D}} + \frac{1}{12}\int d^{-1}\left(df f^{-1}, [df f^{-1}, df f^{-1}]\right)_{\mathcal{D}} + \\ +\frac{1}{2}\int d\sigma d\tau \left(f^{-1}\partial_z f, S^a + i\mathcal{E}_f S^a\right)_{\mathcal{D}} \left(A_f^{-1}\right)_{ab} \left(S^b, f^{-1}\partial_{\bar{z}} f\right)_{\mathcal{D}}. \tag{4.6}$$

In spite of the explicit presence of the imaginary unit in this formula, the action (4.6) is always real.

4.1 T-duality and the anisotropic model

The Yang-Baxter σ-model (3.7) on the compact group $K = \mathrm{SU}(2)$ is left invariant and right Poisson-Lie symmetric. The left invariance can be also interpreted [2] as the (left) Poisson-Lie symmetry corresponding to a trivial (null) Poisson bracket on K. The dual group is then the dual space $\mathcal{K}^*$ of $\mathcal{K} = Lie(K)$ and the group law of $\mathcal{K}^*$ is just the addition of vectors. The Lie algebra of the dual group $\mathcal{K}^*$ is thus clearly abelian.

So the Yang-Baxter σ-model can be dualized with respect to the left $\mathcal{K}^*$-Poisson-Lie symmetry. The dual σ-model has the well-known action [1, 2]:

$$S = \frac{1}{2}\int d\sigma d\tau \partial_+ \chi^a M_{ab}(\chi)\partial_- \chi^b . \tag{4.7}$$

Here χ^a are the coordinates on the target $\mathcal{K}^*$ with respect to some chosen basis $t^*_a \in \mathcal{K}^*$. The matrix $M(\chi)$ is the inverse of

$$\left(M^{-1}\right)^{ab} = \left(\tilde{R}\right)^{ab} + f^{ab}{}_c \chi^c , \tag{4.8}$$

where $f^{ab}{}_c$ are the structure constants of $\mathcal{K} = Lie(K)$ in the basis T^a dual to t^*_a and $(\tilde{R})^{ab} = (T^a, \tilde{R}T^b)_{\mathcal{D}}$. We note that the duality between (3.7) and (4.7) is also referred to as the traditional non-abelian T-duality [1].

The dualization of the Yang-Baxter σ-model with respect to the right Poisson-Lie symmetry gives the model (4.1). Note, however, that the operator $\mathcal{E}$, though being K-invariant, need not be AN-invariant. In other words, the dual model need not be Yang-Baxter from the point of view of the dual group $\tilde{G}$. In any case, the action of the dual model can be rewritten in the following nice way [4]:

$$S = \frac{1}{2}\int d\sigma d\tau \left(\partial_+ b b^{-1}, \left(\tilde{R} + \tilde{\Pi}_b\right)^{-1} \partial_- b b^{-1}\right)_{\mathcal{D}}. \tag{4.9}$$

Here $b(\tau, \sigma)$ is the map from the worldsheet into the group $\tilde{G} = AN$ and the operator $\tilde{\Pi}_b : \mathcal{K} \to \tilde{\mathcal{G}}$ is given by

$$\tilde{\Pi}_b = \tilde{\mathcal{J}} Ad_b \mathcal{J} Ad_{b^{-1}} \mathcal{J} . \tag{4.10}$$

We realize (cf.$\tilde{(}$2.15) and (2.16)) that $\tilde{\Pi}_b$ defines the Poisson-Lie bracket on $\tilde{G}$. It turns out also that the dual σ-model is right G-Poisson-Lie symmetric with respect to the right action of $\tilde{G}$ on itself [2, 4].

We thus observe that the Yang-Baxter σ-model (3.7) living on K has two different duals (4.7) and (4.9) living respectively on the groups $\mathcal{K}^*$ and $\tilde{G} = AN$. We may call this phenomenon the enhanced Poisson-Lie T-duality. The dual σ-models are either of the form (4.9), if we dualize (3.7) with respect to the right AN-Poisson-Lie symmetry (the Drinfeld double is $K^{\mathbb{C}}$), or of the form (4.7) if we dualize with respect to the left $\mathcal{K}^*$-Poisson-Lie symmetry or, in other words, with respect to the traditional non-abelian duality (the Drinfeld double is the cotangent bundle T^*K).

We give for completeness also the explicit description of the dual model (4.9) living on AN. The matrices $\tilde{R}_\alpha$ in (4.9) are given by (3.6) and the Poisson-Lie operator (4.10) is given by

$$\begin{aligned}
\tilde{\Pi}_b T^1 &= u_2 t_3 - [u_1^2 + u_2^2 + \frac{1}{4}e^{4\phi} - \frac{1}{4}]t_2\,;\\
\tilde{\Pi}_b T^2 &= -u_1 t_3 + [u_1^2 + u_2^2 + \frac{1}{4}e^{4\phi} - \frac{1}{4}]t_1\,;\\
\tilde{\Pi}_b T^3 &= u_1 t_2 - u_2 t_1\,.
\end{aligned}$$

Finally, in the parametrization (3.14), we have

$$\left(dbb^{-1}\right)^1 = e^{2\phi} d\left(e^{-2\phi}u_1\right), \qquad \left(dbb^{-1}\right)^2 = e^{2\phi} d\left(e^{-2\phi}u_2\right), \qquad \left(dbb^{-1}\right)^3 = d\phi\,.$$

4.2 T-duality and the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model

Since the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ *WZW* model has the Yang-Baxter property, it can be dualized either with respect to the ordinary left symmetry or with respect to the right Poisson-Lie symmetry. In the former case the Drinfeld double is the cotangent bundle T^*AN of the group AN, which has, up to conjugacy, seven maximally isotropic subalgebras, and in the latter it is $\mathrm{SL}(2,\mathbb{C})$ with three maximally isotropic subalgebras. One such subalgebra of $Lie(T^*AN)$ and one of $sl(2,\mathbb{C})$ give the same $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ *WZW* theory. Thus it remains eight potentially new models.

From the eight cases we must remove one isotropic subalgebra ($Lie(AN)$) of $\mathrm{SL}(2,\mathbb{C})$ and one of T^*AN because the respective matrices A_f are not invertible (cf (4.5)). Thus there remains six σ-models, but one of them is not conformal due to tracefulness of its structure constants (cf. [11] for general discussion of this issue) and other three of them can be obtained also by abelian duality with spectators so we shall not discuss them in any detail. Thus there remains a couple of possibly interesting σ-models, dual in a truly non-Abelain way to the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ *WZW* model. One of them is based on the double T^*AN and the other on $\mathrm{SL}(2,\mathbb{C})$. Here are the details:

4.2.1 The double T^*AN

The group law of AN can be obtained from the matrix multiplication in the parametrization (3.14). Replacing ϕ by λ and $u_{1,2}$ by $L_{1,2}$, the law reads

$$(L_1, L_2, \lambda)(K_1, K_2, \kappa) = \left(L_1 + e^{2\lambda}K_1, L_2 + e^{2\lambda}K_2, \lambda + \kappa\right).$$

The group law on the cotangent bundle T^*AN is the standard one: the semidirect product of AN acting in the coadjoint way on the dual of $Lie(AN)$ viewed as the abelian additive group. The resulting group is six-dimensional, it has the topology $\mathbb{R}^6$, and, in the suitable parametrization, the group law turns out to be

$$(L,\lambda,l,w)(K,\kappa,k,y) = \left(L + e^{2\lambda}K, \lambda+\kappa, l + e^{-2\lambda}k, w + y + 2(L,k)e^{-2\lambda}\right),$$

where $L = (L^1, L^2)$, $K = (K^1, K^2)$ and $(K, L) = K^1L^1 + K^2L^2$. We have

$$M^{-1} \equiv (L,l,w,\lambda)^{-1} = \left(-e^{-2\lambda}L, -\lambda, -e^{2\lambda}l, -w + 2(L,l)\right)$$

and the unit element is $e_d = (0,0,0,0)$.
The set of the left invariant vector fields: The right multiplication by $(\delta L^\alpha, 0, 0, 0)$ induces

$$\nabla^R_{t_\alpha} = e^{2\lambda}\frac{\partial}{\partial L^\alpha};$$

by $(0, 0, \delta l_\alpha, 0)$ gives

$$\nabla^R_{\tau^\alpha} = e^{-2\lambda}\left(\frac{\partial}{\partial l_\alpha} + 2L^\alpha\frac{\partial}{\partial w}\right);$$

and by $(0, 0, 0, \delta w)$ and $(0, \delta\lambda, 0, 0)$ yields respectively

$$\nabla^R_{\tau^0} = \frac{\partial}{\partial w}, \qquad \nabla^R_{t_0} = \frac{\partial}{\partial \lambda}.$$

Hence we obtain the following bracket on $Lie(T^*AN)$:

$$\begin{aligned}
&[t_0, t_\alpha] = 2t_\alpha, \qquad [t_\alpha, t_\beta] = 0, \qquad \left[\tau^i, \tau^j\right] = 0, \qquad i,j = 0,1,2;\\
&\left[t_i, \tau^0\right] = 0, \qquad [t_0, \tau^\alpha] = -2\tau^\alpha, \qquad \left[t_\alpha, \tau^\beta\right] = 2\delta^\beta_\alpha\tau^0, \qquad \alpha,\beta = 1,2.
\end{aligned}$$

There is a non-degenerate symmetric invariant bilinear form on $Lie(T^*AN)$ defined by

$$\left(\tau^i, \tau^j\right)_d = (t_i, t_j)_d = 0, \qquad \left(\tau^i, t_j\right)_d = \delta^i_j. \tag{4.11}$$

It is this form that makes T^*AN the Drinfeld double of AN.

Now we would like to write down the duality invariant first order action (2.6) on the double T^*AN. Such action leads to σ-models which are *right* symmetric or Poisson-Lie symmetric. Since we wish to dualize with respect to the *left* symmetry of our $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model we have to generalize our formalism in order to be able to take into account also the left Poisson-Lie symmetric models. This generalization is very easy: if $L_R(g)$ is the lagrangian of a right Poisson-Lie symmetric σ-model then $L_L(g) = L_R(g^{-1})$ is left Poisson-Lie symmetric.

Thus we can consider the action (2.6) for the double T^*AN and we choose the following operator $\mathcal{E}$:

$$\begin{aligned}
&\mathcal{E}t_1 = -t_2, \qquad \mathcal{E}t_2 = t_1, \qquad \mathcal{E}t_0 = 2\tau^0;\\
&\mathcal{E}\tau^1 = -\frac{1}{2}t_1 + \tau^2, \qquad \mathcal{E}\tau^2 = -\frac{1}{2}t_2 - \tau^1, \qquad \mathcal{E}\tau^0 = -\frac{1}{2}t_0.
\end{aligned} \tag{4.12}$$

One easily checks that $\mathcal{E}$ is selfadjoint with respect to the form (4.11) and it holds $\mathcal{E}^2 = -Id$.

The maximally isotropic subalgebras of $Lie(T^*AN)$ have been classified (up to conjugacy) in [12]. There are two one-parametric families of them and five singular cases. Here is the list

$$
\begin{array}{ll}
1\gamma) & \mathrm{Span}(\tau^0, \gamma_1\tau^1+\gamma_2\tau^2, -\gamma_2 t_1+\gamma_1 t_2)\,, \quad \gamma_1^2+\gamma_2^2=1\,;\\
2\delta) & \mathrm{Span}(t_0, \delta_1 t_1+\delta_2 t_2, -\delta_2\tau^1+\delta_1\tau^2)\,, \quad \delta_1^2+\delta_2^2=1\,;\\
3) & \mathrm{Span}(t_0, \tau^2+t_1, \tau^1-t_2)\,; \qquad 4) \quad \mathrm{Span}(t_0, \tau^2-t_1, \tau^1+t_2)\,;\\
5) & \mathrm{Span}(\tau^0,\tau^1,\tau^2)\,; \qquad 6) \quad \mathrm{Span}(\tau^0, t_1, t_2) \qquad 7) \quad \mathrm{Span}(t_0,t_1,t_2)\,.
\end{array}
$$

First of all, the choice of the subalgebra 5) leads to the σ-model (4.6) living on the target $T^*AN/Lie^*(AN) = AN$. Performing on it the transformation $f \to f^{-1}$ leading from the right to the left symmetric model, we obtain precisely the action (3.16) of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZWmodel. The choices 7) leads to a nonconformal σ-model since the structure constants are tracefull. The possibility 6) must be also rejected since the matrix (4.5) is not invertible in this case. The choices 1γ), 3) and 4) lead to σ-models which can be also obtained by using the standard abelian T-duality with respect to the isometries of the $u, \bar{u}$ plane (cf.\~(3.16)). We do not detail the corresponding σ-model metrics and H-fields here.

Thus the only truly non-abelian case is 2δ). It turns out that all choices of δ gives the same σ-model, hence we take $\delta_1 = 1, \delta_2 = 0$. The corresponding subgroup will be denoted as $C = \exp\left(\mathrm{Span}(t_0, t_1, \tau^2)\right)$. The σ-model target is then T^*AN/C and can be identified with the group $\tilde{C} = \exp\left(x_0\tau^0 + x_1\tau^1 + x_2 t_2\right)$, where x_j are coordinates on the group manifold $\tilde{C}$. Note that C is the Poincaré group in two dimension and $\tilde{C}$ is abelian. Inserting a decomposition $l = \tilde{c}c$ ($c \in C, \tilde{c} \in \tilde{C}$) in the first order action (2.6) and eliminating the variable c gives the following σ-model action:

$$S = \frac{1}{2}\int d\sigma d\tau \partial_z x_k N^{kl}(x)\partial_{\bar{z}} x_l\,,$$

where

$$(N^{-1})_{kl} = \begin{pmatrix} -2 & 2ix_1 & -2ix_2 \\ -2ix_1 & 0 & 1 \\ 2ix_2 & 1 & \frac{1}{2} \end{pmatrix}. \tag{4.13}$$

In particular, the metric part of the σ-model background is

$$ds^2 = \frac{-dx_0^2 - (1+4x_2^2)dx_1^2 - 4x_1^2 dx_2^2 + (2-4x_1x_2)dx_1 dx_2}{2 - 2x_1^2 - 8x_1x_2}\,.$$

This metric changes signature (in some domain of the space-time it is riemannian in another pseudo-riemannian). It is the matter of taste to say that it is either its very interesting or rather pathological feature. We adopt the latter (and more coservative) viewpoint and turn to the investigation of the double $\mathrm{SL}(2,\mathbb{C})$.

4.2.2 The double $\mathrm{SL}(2,\mathbb{C})$: dS_3/AdS_3 duality

There are (up to conjugacy) three maximally isotropic subalgebras of $sl(2,\mathbb{C})$: $su(2)$, $sl(2,\mathbb{R})$ and $Lie(AN)$. For $Lie(AN)$, the matrix A_f (cf.\~(4.5)) is not invertible, hence it remains the T-duality between two σ-models living respectively on the cosets $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$

$= AN$ and $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathbb{R})$. The former model is just the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model with the action (3.16), describing strings in the euclidean AdS_3 target and we shall see soon that the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathbb{R})$ model captures the dynamics of strings in the three-dimensional de Sitter space dS_3. We recall that here the Poisson-Lie T-duality is euclidean, or, in other words, it relates two σ-models with *real* euclidean action. This means that the de Sitter H-field is imaginary.

The action of the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathbb{R})$ model is of course given by the general formula (4.6). To make it more explicit we have to specify the sections f of the fibration $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathbb{R})$. This requires some preliminary exposition of the geometry of this coset:

For convenience, we first embed $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(2,\mathbb{C})$ in a nonstandard way

$$\begin{pmatrix} \mu & i\nu \\ i\rho & \lambda \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})\,, \qquad \mu,\nu,\rho,\lambda \in \mathbb{R}\,, \qquad \mu\lambda + \nu\rho = 1\,. \tag{4.14}$$

We shall refer to the atypically embedded group $\mathrm{SL}(2,\mathbb{R})$) as to $\mathrm{SL}^a(2,\mathbb{R})$. We recall that σ-models (4.6) on D/M and D/M' have the same target geometry if M and M' are conjugated in D. In fact, our embedding $\mathrm{SL}^a(2,\mathbb{R})$ is conjugated to the standard one (real matrices with unit determinant) by the following element of $\mathrm{SL}(2,\mathbb{C})$:

$$\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ i-1 & 1-i \end{pmatrix}.$$

Consider a space dS_3 of hermitean $2d$-matrices with determinant equal to (-1). They can be written as

$$s = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix}, \qquad uv - \bar{w}w = -1\,. \tag{4.15}$$

Clearly, dS_3 is nothing but the de Sitter space, if it is equipped with the Minkowski metric

$$ds^2_{dS} = dudv - d\bar{w}dw\,,$$

restricted to the surface $uv - \bar{w}w = -1$. There exists a natural map $s : \mathrm{SL}(2,\mathbb{C}) \to dS_3$ defined as

$$s(l) = l\sigma_1 l^\dagger\,, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that the preimage of the point $\sigma_1 \in dS_3$ under the map s is a group, in fact, it is our group $\mathrm{SL}^a(2,\mathbb{R})$. Thus we see that the map s defines also an injective map from the coset $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}^a(2,\mathbb{R})$ to the de Sitter space dS_3. By a small abuse of notation we call this map also s. It is not difficult to prove that the map $s : \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}^a(2,\mathbb{R}) \to dS_3$ is also surjective.

Our next goal is to find the section f of the fibration $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}^a(2,\mathbb{R})$ to be inserted in the action (2.6). To do this, we consider first the Iwasawa decomposition of $\mathrm{SL}^a(2,\mathbb{R})$:

$$\begin{pmatrix} \mu & i\nu \\ i\rho & \lambda \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & iM \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix}, \phi, M \in \mathbb{R}, 0 \le \theta \le 2\pi\,,$$

and the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & iM \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix}, \tag{4.16}$$

where $a,b,c,d \in \mathbb{C}$, $\phi, L, M \in \mathbb{R}$ and $\alpha,\beta \in \mathbb{C}$ with $\alpha\bar\alpha + \beta\bar\beta = 1$. We recognize in the first matrix on the r.h.s. of (4.16) an element of the group SU(2). Comparing those two Iwasawa decompositions, we arrive at the conclusion, that we can choose f within the form $\begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$. We remark the following identity

$$\begin{pmatrix} \cos\theta + i\frac{L}{\sqrt{L^2+1}}\sin\theta & i\frac{1}{\sqrt{L^2+1}}\sin\theta \\ i\frac{1}{\sqrt{L^2+1}}\sin\theta & \cos\theta - i\frac{L}{\sqrt{L^2+1}}\sin\theta \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & i\sqrt{L^2+1}\sin\theta \\ i\frac{1}{\sqrt{L^2+1}}\sin\theta & \cos\theta \end{pmatrix}. \tag{4.17}$$

The second matrix on the r.h.s. of (4.17) is clearly in $\mathrm{SL}^a(2,\mathbb{R})$, hence the product of two matrices on the l.h.s. is in the same coset as $\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$. Moreover, the first matrix on the l.h.s. of (4.17) is in SU(2). We call it $R_L(\theta)$ and we note that

$$R_L(\theta_1+\theta_2) = R_L(\theta_1)R_L(\theta_2)\,.$$

For θ close to zero it holds

$$R_L(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\frac{\theta}{\sqrt{L^2+1}} \begin{pmatrix} L & 1 \\ 1 & -L \end{pmatrix} + O(\theta^2) \tag{4.18}$$

and it is true also that

$$Tr\left\{ \begin{pmatrix} L & 1 \\ 1 & -L \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} = 0\,. \tag{4.19}$$

The facts (4.18) and (4.19) imply that there is an L-dependent global gaussian decomposition of SU(2):

$$\begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} = \begin{pmatrix} \cos\vartheta + i\frac{L}{\sqrt{L^2+1}}\sin\vartheta & i\frac{1}{\sqrt{L^2+1}}\sin\vartheta \\ i\frac{1}{\sqrt{L^2+1}}\sin\vartheta & \cos\vartheta - i\frac{L}{\sqrt{L^2+1}}\sin\vartheta \end{pmatrix} \times$$
$$\times \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \cos\theta + i\frac{L}{\sqrt{L^2+1}}\sin\theta & i\frac{1}{\sqrt{L^2+1}}\sin\theta \\ i\frac{1}{\sqrt{L^2+1}}\sin\theta & \cos\theta - i\frac{L}{\sqrt{L^2+1}}\sin\theta \end{pmatrix},$$

where $0 \le \vartheta, \theta \le \pi$ and $0 \le \chi \le \pi/2$. Regarding the identity (4.17), we find immediately the needed section f:

$$f = \begin{pmatrix} \cos\vartheta + i\frac{L}{\sqrt{L^2+1}}\sin\vartheta & i\frac{1}{\sqrt{L^2+1}}\sin\vartheta \\ i\frac{1}{\sqrt{L^2+1}}\sin\vartheta & \cos\vartheta - i\frac{L}{\sqrt{L^2+1}}\sin\vartheta \end{pmatrix} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \tag{4.20}$$

where $0 \le \vartheta \le \pi$, $0 \le \chi \le \pi/2$ and $L \in \mathbb{R}$.
The map $s(f)$ now gives the transformation between the "cylindrical" coordinates ϑ, χ, L and the de Sitter parametrization (4.15):

$$\frac{1}{2}(u+v) = L, \quad \frac{1}{2}(u-v) = L\cos 2\chi + \sin 2\chi \cos 2\vartheta\,,$$
$$w = \cos 2\chi - L\sin 2\chi\cos 2\vartheta - i\sqrt{L^2+1}\sin 2\chi \sin 2\vartheta\,. \tag{4.21}$$

The reader may verify that, indeed, it holds $uv - \bar{w}w = -1$. Now we read off from (4.6) the metric and the three-form field H of the σ-model on $\mathrm{SL}(2,\mathbf{C})/\,\mathrm{SL}^a(2,\mathbb{R})$:

$$\begin{aligned} ds^2 &= (f^{-1}df, S^a)_{\mathcal{D}}(A_f^{-1})_{ab}(S^b, f^{-1}df)_{\mathcal{D}}\,; \\ H &= +\frac{i}{2}d\left\{\left(f^{-1}df, \mathcal{E}_f S^a\right)_{\mathcal{D}} \wedge \left(A_f^{-1}\right)_{ab}\left(S^b, f^{-1}df\right)_{\mathcal{D}}\right\} - i\frac{1}{12}\left(dff^{-1} \stackrel{\wedge}{,} [dff^{-1} \stackrel{\wedge}{,} dff^{-1}]\right)_{\mathcal{D}}. \end{aligned}$$

We choose the following basis of the $Lie(\mathrm{SL}^a(2,\mathbb{R}))$:

$$S^+ = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \qquad S^- = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \qquad S^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that the operator $\mathcal{E} = \mathcal{I}$ is $\mathrm{SL}(2,\mathbb{C})$-invariant, hence $\mathcal{E}_f = \mathcal{I}$ and the matrix A_f (cf.~(4.5)) is also independent on f. The non-zero components of its inverse are

$$(A_f^{-1})_{+-} = (A_f^{-1})_{-+} = -1\,, \qquad (A_f^{-1})_{00} = 2\,.$$

Considering (4.20), it is straightforward to calculate the form $f^{-1}df$ and hence the metric and the H-field of the $\mathrm{SL}(2,\mathbf{C})/\,\mathrm{SL}^a(2,\mathbb{R})$ model:

$$\begin{aligned} ds^2 &= -\frac{1}{2}\frac{(dL)^2}{L^2+1}\sin^2 2\chi \sin^2 2\vartheta + 2(L^2+1)(d\chi^2 + \sin^2 2\chi d\vartheta^2) + \\ &\quad +2\cos 2\vartheta dL d\chi - \sin 4\chi \sin 2\vartheta dL d\vartheta\,; \qquad (4.22) \\ H &= -4i\sqrt{L^2+1}\sin 2\chi dL \wedge d\chi \wedge d\vartheta\,. \qquad (4.23) \end{aligned}$$

By using (4.21), we can check easily that (up to a normalization factor 1/2) the metric (4.22) is nothing but the standard de Sitter metric (4.2.2).

Because the Lie algebra $sl(2,\mathbb{R})$ does not have the traceful structure constants (i.e. ${f^{ab}}_b = 0$), we expect to find a dilaton Φ such that the background (4.22, (4.23) will satisfy the following conditions [13, 14] of the conformal invariance

$$R_{mn} - \frac{1}{4}H_{mpq}H_n^{\ pq} + 2D_m D_n \Phi = 0\,, \qquad -\frac{1}{2}D_n H^n_{\ pq} + \partial_n \Phi H^n_{pq} = 0\,. \qquad (4.24)$$

Here R_{mn} means the Ricci tensor and D_n the covariant derivative. The computation indeed shows that the conditions (4.24) are satisfied for $\Phi = 0$.

4.2.3 T-duality and loop group symmetry

It is well-known that the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW model has a $\mathrm{SL}(2,\mathbb{C})$ current symmetry. It can be very transparently understood in the first order formalism, where the action reads

$$S(l) = \frac{1}{2}\int d\sigma d\tau \mathrm{ImTr}\left(\partial_{\bar{z}} ll^{-1}\partial_\sigma ll^{-1}\right) + \frac{1}{12}\int d^{-1}\mathrm{ImTr}\left(dll^{-1} \wedge \left[dll^{-1} \stackrel{\wedge}{,} dll^{-1}\right]\right). \quad (4.25)$$

By using the Polyakov-Wiegmann formula [15]

$$S(l_1 l_2) = S(l_1) + S(l_2) + \int d\sigma d\tau \mathrm{ImTr}\left(l_1^{-1}\partial_{\bar{z}} l_1 \partial_\sigma l_2 l_2^{-1}\right), \qquad (4.26)$$

the field equations of the first order model (4.25) can now be easily obtained. We set in (4.26) $l_1 = l$ and $l_2 = 1 + \chi(z,\bar{z})$, with $\chi \in sl(2,\mathbf{C})$ being infinitesimal and we find

$$\delta S = \int d\sigma d\tau \mathrm{ImTr}\left(l^{-1}\partial_{\bar{z}} l \partial_\sigma \chi\right).$$

Since $\partial_\sigma \chi$ is arbitrary and the form ImTr non-degenerate, we conclude that the first order field equations have extremely simple form:

$$l^{-1}\partial_{\bar{z}} l = 0. \tag{4.27}$$

The symmetries of the first order equation (4.27) are evident: a solution $l(z)$ becomes clearly another solution upon the transformation

$$l(z) \to g(z)l(z), \qquad g(z) \in SL(2,\mathbb{C}). \tag{4.28}$$

Our next goal is to visualize the huge symmetry (4.28) in the second order $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZWmodel and its $\mathrm{SL}(2,\mathbf{C})/\mathrm{SL}^a(2,\mathbb{R})$ de Sitter dual. 1. $\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$ model. Recall that the target space is the group AN and the lagrangian is given by (3.16). We can perform the change of variables on the target AN as follows

$$h = bb^\dagger = \begin{pmatrix} e^{2\phi} + 4e^{-2\phi}u\bar{u} & 2e^{-2\phi}\bar{u} \\ 2e^{-2\phi}u & e^{-2\phi} \end{pmatrix}.$$

In terms of the matrix h, the σ-model background (3.17) can be rewritten as

$$ds^2_{AdS} = \frac{1}{8}\mathrm{Tr}\left(dhh^{-1}dhh^{-1}\right), \tag{4.29}$$

$$H = \frac{1}{48}\mathrm{Tr}\left(dhh^{-1} \wedge \left[dhh^{-1} \stackrel{\wedge}{,} dhh^{-1}\right]\right). \tag{4.30}$$

In other words, the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZWaction (3.16) can be rewritten as

$$S = i\int dz \wedge d\bar{z}\mathrm{Tr}\left(\partial_z hh^{-1}\partial_{\bar{z}} hh^{-1}\right) + \frac{i}{6}\int d^{-1}\mathrm{Tr}\left(dhh^{-1} \wedge \left[hh^{-1} \stackrel{\wedge}{,} dhh^{-1}\right]\right). \tag{4.31}$$

Note that in (4.29), (4.30) and (4.31) there is the full trace, not only its imaginary part. How the loop symmetry (4.28) manifests itself from the point of view of the second order action (4.31)? First note that from the first order trajectory l we obtain the second order trajectory b by the Iwasawa decomposition $l = bk$, $b \in AN$, $k \in \mathrm{SU}(2)$ (cf.$\tilde{(}$3.11)). This means that

$$h = ll^\dagger$$

and the symmetry (4.28) becomes

$$h(z,\bar{z}) \to g(z)h(z,\bar{z})g^\dagger(\bar{z}). \tag{4.32}$$

Indeed, by the direct use of the corresponding Polyakov-Wiegmann formula, we see that the model (4.31) has the symmetry (4.32).

1. $\mathrm{SL}(2,\mathbf{C})/\,\mathrm{SL}^a(2,\mathbb{R})$ model. Recall that the target is the space dS_3 of hermitean matrices with determinant (-1):

$$s = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix}, \qquad uv - w\bar{w} = -1\,.$$

Using the parametrization (4.21) and ignoring an unimportant overall normalization, the σ-model metric and H-field (4.22), (4.23) can be rewritten as

$$\begin{aligned} ds^2_{dS} &= \mathrm{Tr}\left(dss^{-1}dss^{-1}\right), \\ H &= \frac{1}{6}\mathrm{Tr}\left(dss^{-1}\wedge\left[dss^{-1} \overset{\wedge}{,} dss^{-1}\right]\right). \end{aligned} \tag{4.33}$$

In other words, the de Sitter σ-model action can be written as

$$S = i\int dz\wedge d\bar{z}\mathrm{Tr}\left(\partial_z ss^{-1}\partial_{\bar{z}}ss^{-1}\right) + \frac{i}{6}\int d^{-1}\mathrm{Tr}\left(dss^{-1}\wedge\left[dss^{-1} \overset{\wedge}{,} dss^{-1}\right]\right). \tag{4.34}$$

Note that the only distiction of the AdS_3 action (4.31) and its dS_3 dual (4.34) is the sign of the respective determinants of the hermitean matrices h and s.
How the loop symmetry (4.28) manifests itself from the point of view of the second order action (4.34)? First note that from the first order trajectory l we obtain the second order trajectory f by the decomposition $l = fm$, $m \in \mathrm{SL}^a(2,\mathbb{R})$ (cf.$\tilde{(}$4.2)). This means that

$$s = f\sigma_1 f^\dagger = l\sigma_1 l^\dagger.$$

and the symmetry (4.28) becomes

$$s(z,\bar{z}) \to g(z)s(z,\bar{z})g^\dagger(\bar{z})\,. \tag{4.35}$$

Indeed, by the direct use of the Polyakov-Wiegmann formula, we see that the model (4.34) has the symmetry (4.35).

5. Conclusions and outlook

We have shown that the simultaneous presence of the ordinary (left) symmetry and the Poisson-Lie (right) symmetry of σ-models on group targets results in a rich T-duality picture. Our main example of a unitary theory was the anisotropic principal chiral model (3.7) and its duals (4.7) and (4.9). However, it seems that a better candidate for settling the quantum status of Poisson-Lie T-duality would be the dual pair of AdS_3 and dS_3 models (4.31) and (4.34). There is a surprising feature of this duality, namely, the dS_3 background has a different signature than the (euclidean) AdS_3 one. Further investigations are needed to elucidate this fact.

In the subsequent paper, we plan to describe how the dS_3/AdS_3 T-duality works for open strings and D-branes.

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